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Uniqueness of Renormalized Solutions of Degenerate Elliptic–Parabolic Problems

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We consider a general class of degenerate elliptic–parabolic problems associated with the equation $b(v)_t = \operatorname{div} a(v, Dv) + f$. Using Kruzhkov's method of doubling variables both in space and time we prove uniqueness and a comparison principle in L^1 for renormalized solutions.

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary if $N \geqslant 2$, $T > 0$. Consider the quasi-linear elliptic–parabolic problem

$$(\mathrm{EP})(v_0, f) \begin{cases} b(v)_t = \operatorname{div} a(v, Dv) + f & \text{on} \quad Q =]0, T[\times \Omega \\ v = 0 & \text{on} \quad \Sigma =]0, T[\times \partial\Omega, \\ b(v)(0, \cdot) = b(v_0) & \text{on} \quad \Omega \end{cases}$$

where

(H1) $b: \mathbb{R} \to \mathbb{R}$ is a continuous, non-decreasing function, satisfying the normalization condition $b(0) = 0$;

(H2) $a: \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous vector field satisfying, for some $1 < p < \infty$, the growth condition

$$|a(k, \xi)| \leqslant C(|k|)(1 + |\xi|^{p-1}) \qquad \text{for all} \quad (k, \xi) \in \mathbb{R} \times \mathbb{R}^N \tag{1}$$

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with $C: \mathbb{R}^+ \to \mathbb{R}^+$ nondecreasing;

(H3) $f \in L^1(Q)$ and $v_0: \Omega \to \overline{\mathbb{R}}$ is measurable with $u_0 = b(v_0) \in L^1(\Omega)$.

Uniqueness of weak solutions of (EP) is known only under restrictive additional assumptions, in particular the assumption of Hölder continuity of the vector field a in the first variable (cf. [1, 19]; see also [4]). In this paper we prove uniqueness and a comparison result for weak solutions and, more generally, renormalized solutions of (EP) under the assumptions (H1)–(H3) and some general additional condition on a satisfied by a large class of vector fields, in particular fields containing a merely continuous convection term (see condition (8) in Section 2). As usual, a weak solution of (EP) is a function $v \in L^p(0, T; W_0^{1,p}(\Omega))$ with $b(v) \in L^1(Q)$, $a(v, Dv) \in (L^{p'}(Q))^N$, satisfying

$$\int_Q \zeta_t b(v) + \zeta f + \int_\Omega \zeta(0, \cdot)\, b(v_0) = \int_Q a(v, Dv) \cdot D\zeta \tag{2}$$

for all $\zeta \in \mathscr{D}([0, T[\times \Omega)$. Existence of weak solutions requires additional regularity of the data u_0 and f (as $u_0 = b(v_0)$ with $B(v_0) = \int_0^{v_0} r\, db(r) \in L^1(\Omega)$, $f \in L^{p'}(0, T; W^{-1,p'}(\Omega))$; see, e.g., [1, 4]). In order to be able to solve (EP) for general L^1-data one needs a more general notion of solution. In this paper we will use the notion of renormalized solution.

As usual, for $k > 0$, we denote by T_k the truncation function defined by

$$T_k(v) = \begin{cases} k & \text{if } \; v > k \\ v & \text{if } \; |v| \leqslant k. \\ -k & \text{if } \; v < -k \end{cases}$$

Definition 1.1. A *renormalized solution* of $(\mathrm{EP})(v_0, f)$ is a measurable function $v: Q \to \mathbb{R}$ satisfying

(i) $b(v) \in L^1(Q)$;

(ii) $T_k(v) \in L^p(0, T; W_0^{1,p}(\Omega))$ for any $k > 0$;

(iii) for all $h \in C_c^1(\mathbb{R})$, $\zeta \in \mathscr{D}([0, T[\times \Omega)$,

$$\int_Q \zeta_t \int_{v_0}^{v} h(r)\, db(r) + \zeta f h(v) = \int_Q a(v, Dv) \cdot D(h(v)\zeta) \tag{3}$$

and, moreover,

$$\int_{Q \cap \{n \leqslant |v| \leqslant n+1\}} (a(v, Dv) - a(v, 0)) \cdot Dv \to 0 \qquad \text{as} \quad n \to \infty. \tag{4}$$

Remark 1.2. (i) Note that in (3) and (4) each term is well-defined. Indeed, the first member of (3) is well-defined as $|\int_{v_0}^{v} h(r)\, db(r)| \leqslant \|h\|_\infty \times |b(v) - b(v_0)|$ and $b(v) \in L^1(Q)$, $b(v_0) \in L^1(\Omega)$. The term on the right-hand side of (3) has to be understood as

$$\int_{Q \cap \{|v| < k\}} a(v, DT_k(v)) \cdot D(h(T_k(v))\zeta) \tag{5}$$

for $k > 0$ such that $\operatorname{supp} h \subset [-k, k]$. Indeed, if $\operatorname{supp} h \subset [-k, k]$, then $h(v) = h(T_k(v))$ and $h(v) = 0$ a.e. on $\{|v| \geqslant k\}$. Since $T_k(v) \in L^p(0, T; W_0^{1,p}(\Omega))$, it is the same for $h(v)\zeta$, and $D(h(v)\zeta) = 0$ a.e. on $\{|v| \geqslant k\}$; due to the growth condition (1), $a(v, DT_k(v))\, \chi_{\{|v| < k\}}$ is in $L^{p'}(Q)^N$ such that the integral (5) is well-defined. Finally, note that $DT_k(v) = DT_l(v)$ a.e. on $\{|v| < k \wedge l\}$, hence the integral (5) is independent of the choice of k satisfying $\operatorname{supp} h \subset [-k, k]$. Similarly, the integral in (4) has to be understood as

$$\int_{Q \cap \{n < |v| < n+1\}} a(v, DT_{n+1}(v)) - a(v, 0)) \cdot DT_{n+1}(v), \tag{6}$$

which is meaningful by the assumptions on a and v. Throughout the paper we use the integral in (3) and (4) only as a notation for the corresponding integral (5) and (6) while we could make more precise the notation Dv (cf. [3]).

(ii) Note that if v is a renormalized solution, then $B_h(v) := \int_0^v h(r)\, db(r) \in L^1(Q)$, $B_h(v)_t \in L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^1(Q)$ and $B_h(v)(0, \cdot) = \int_0^{v_0} h(r)\, db(r)$ in $\mathscr{D}'(\Omega)$.

(iii) By approximation, (3) holds for any $h \in W^{1,\infty}(\mathbb{R})$ with compact support and all $\zeta \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ with $\zeta_t \in L^\infty(Q)$, $\zeta(T, \cdot) = 0$.

The notion of renormalized solution is an extension of the notion of weak solution:

PROPOSITION 1.3. (i) *A weak solution of* (EP) *is a renormalized solution. More precisely, let* $v \in L^p(0, T, W_0^{1,p}(\Omega))$ *with* $b(v) \in L^1(Q)$ *and* $a(v, Dv) \in L^{p'}(Q)^N$. *Then v is a weak solution of* (EP) *if and only if v is a renormalized solution of* (EP).

(ii) *Let* $v \in L^\infty(Q)$. *Then v is a weak solution of* (EP) *if and only if v is a renormalized solution.*

The proof of this result is based on the following "integration-by-parts"-formula:

LEMMA 1.4. *Let* $b\colon \mathbb{R} \to \mathbb{R}$ *be continuous and non-decreasing with* $b(0)=0$, $v \in L^p(0, T; W_0^{1,p}(\Omega))$ *with* $b(v) \in L^1(Q)$, $b(v)_t \in L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^1(Q)$, *and* $b(v)(0, \cdot) = b(v_0)$, *where* $v_0\colon \Omega \to \overline{\mathbb{R}}$ *is measurable with* $b(v_0) \in L^1(\Omega)$. *Then*

$$-\int_0^T \langle b(v)_t, h(v)\xi \rangle \, dt = \int_Q \xi_t \int_{v_0}^{v} h(r) \, db(r) \, dx \, dt \tag{7}$$

for any pair $h \in W^{1,\infty}(\mathbb{R})$, $\xi \in W^{1,\infty}(Q)$ *with* $\xi(T)=0$ *and* $h(v)\xi \in L^p(0, T; W_0^{1,p}(\Omega))$. ($\langle \cdot, \cdot \rangle$ *being the duality pairing between* $W^{-1,p'}(\Omega) + L^1(\Omega)$ *and* $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$).

This version may be proved in essentially the same way as the classical "integration-by-parts" formula of [1], respectively, the generalizations considered in [11, 19]. For the sake of completeness a proof is given in the Appendix, where the reader may also find a proof of Proposition 1.3.

Remark 1.5. The notion of renormalized solution or similar notions have been introduced in the past decade for different problems and various existence and uniqueness results have been obtained (see, for instance, [3, 7, 8, 14, 18, 20]). In this paper, we are not concerned with existence of renormalized solutions of (EP) (see [22] for this problem). In [4], existence of mild solutions of (EP) in the sense of nonlinear semigroup theory has been shown under the additional assumption that a is monotone in $\xi \in \mathbb{R}^N$ and satisfies a coerciveness condition,

$$(a(r, \xi) - a(r, 0)) \cdot \xi + C(b(r)) \geqslant c(b(r)) \, |\xi|^p,$$

for some continuous functions $c, C\colon \mathbb{R} \to]0, \infty[$. Under some additional assumptions, one can prove that mild solutions are weak or, more generally, renormalized solutions (cf. [22]; see also [5]).

This paper is organized as follows: in Section 2 we state and comment on the main result, the uniqueness theorem for the initial boundary value problem (EP). Section 3 is devoted to the proof of this theorem. In Section 4 we discuss possible extensions. The proofs of Proposition 1.3 and Lemma 1.4 are given in the Appendix.

2. THE MAIN RESULT

In order to prove uniqueness of renormalized solutions of (EP) we assume that a satisfies the additional condition

$$\begin{aligned}(a(r, \xi) - a(s, \eta)) \cdot (\xi - \eta) + C(r, s)(1 + |\xi|^p + |\eta|^p) \, |r - s| \\ \geqslant \Gamma(r, s) \cdot \xi + \hat{\Gamma}(r, s) \cdot \eta\end{aligned} \tag{8}$$

for all $r, s \in \mathbb{R}$, $\xi, \eta \in \mathbb{R}^N$, where $C: \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$, $\Gamma, \hat{\Gamma}: \mathbb{R} \times \mathbb{R} \to \mathbb{R}^N$ are continuous.

Remark 2.1. (i) Inequality (8) implies $\Gamma(r, r) = \hat{\Gamma}(r, r) = 0$ for all $r \in \mathbb{R}$. Indeed, choosing $r = s$, $\eta = 0$, $\xi = tv$, $t > 0$, $v \in \mathbb{R}^N$ in (8), we get $t(a(r, tv) - a(r, 0)) \cdot v \geqslant t\,\Gamma(r, r) \cdot v$. Dividing by t and passing with $t \to 0$, we find $0 \geqslant \Gamma(r, r) \cdot v$ for all $v \in \mathbb{R}^N$, hence $\Gamma(r, r) = 0$. Using the same arguments we obtain the corresponding result for $\hat{\Gamma}$.

(ii) Inequality (8) implies that a is monotone in $\xi \in \mathbb{R}^N$. This follows immediately from the preceding remark.

Remark 2.2. Note that, if $a(r, \xi)$ satisfies (8), then, for any $F: \mathbb{R} \to \mathbb{R}^N$, the same is true for $a(r, \xi) + F(r)$, $\Gamma, \hat{\Gamma}$ being replaced by $\Gamma(r, s) + F(r) - F(s)$, $\hat{\Gamma}(r, s) + F(s) - F(r)$, respectively. In particular, let $a_0: \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ satisfy Alt–Luckhaus-type assumptions (cf. [1])

$$(a_0(r, \xi) - a_0(r, \eta)) \cdot (\xi - \eta) \geqslant c_0(|r|)\, |\xi - \eta|^p$$

$$|a_0(r, \xi) - a_0(s, \xi)| \leqslant C_0(|r| + |s|)(1 + |\xi|^{p-1})\, |r - s|^{1/p'}$$

$$|a_0(r, \xi)| \leqslant M_0(1 + |\xi|^{p-1}),$$

where $c_0: \mathbb{R}^+ \to]0, \infty[$ is non-increasing, $C_0: \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, and $M_0 > 0$. Then, for any $F: \mathbb{R} \to \mathbb{R}^N$ continuous, $a(r, \xi) = a_0(r, \xi) + F(r)$ satisfies (1) and (8): this is clear for (1); as to (8), using the remark above, it is sufficient to note that, by Young's inequality,

$$\begin{aligned}
&(a_0(r, \xi) - a_0(s, \eta)) \cdot (\xi - \eta) \\
&\quad \geqslant c_0(|r|)\, |\xi - \eta|^p - C_0(|r| + |s|)(1 + |\eta|^{p-1})\, |r - s|^{1/p'}\, |\xi - \eta| \\
&\quad \geqslant (c_0(|r|) - \varepsilon)\, |\xi - \eta|^p - C_\varepsilon(r, s)(1 + |\eta|^p)\, |r - s| \\
&\quad \geqslant -C_\varepsilon(r, s)(1 + |\eta|^p)\, |r - s|
\end{aligned}$$

for $\varepsilon > 0$ sufficiently small, for some continuous function $C_\varepsilon: \mathbb{R}^2 \to \mathbb{R}^+$.

Let us now state our main result. As usual, sign^+ denotes the multi-valued function defined by $\text{sign}^+(r) = 0$ if $r < 0$, $\text{sign}^+(0) = [0, 1]$, $\text{sign}^+(r) = 1$ if $r > 0$ and we denote by sign_0^+ its single-valued section which takes the value 0 in $r = 0$.

THEOREM 2.3. *Assume that* (H1)–(H2) *and the additional condition* (8) *hold. For* $i = 1, 2$, *let* $v_{0i}: \Omega \to \overline{\mathbb{R}}$ *be measurable with* $b(v_{0i}) \in L^1(\Omega)$,

$f_i \in L^1(Q)$. Let v_i be a renormalized solution of (EP)(v_{0i}, f_i), $i=1,2$. *Then there exists $\kappa \in \operatorname{sign}^+(v_1 - v_2)$ such that, for a.e.* $0<t<T$,

$$\int_\Omega (b(v_1)(t) - b(v_2)(t))^+ \leqslant \int_\Omega (b(v_{01}) - b(v_{02}))^+ + \int_0^t \int_\Omega \kappa(f_1 - f_2). \quad (9)$$

In particular, for any $v_0 \colon Q \to \overline{\mathbb{R}}$ measurable with $b(v_0) \in L^1(\Omega)$, $f \in L^1(Q)$, there is uniqueness of $u = b(v)$ for v renormalized solution of (EP)(v_0, f).

Remark 2.4. By Theorem 2.3 we also have uniqueness of renormalized solutions of the corresponding stationary problem

$$(\mathrm{E})(f) \begin{cases} b(v) - \operatorname{div} a(v, Dv) = f & \text{on } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where, for $f \in L^1(\Omega)$, a renormalized solution of (E)(f) is a measurable function $v \colon \Omega \to \overline{\mathbb{R}}$ with $b(v) \in L^1(\Omega)$, $T_k(v) \in W_0^{1,p}(\Omega)$ for all $k>0$ satisfying

$$\int_\Omega b(v)\, h(v)\phi + a(v, Dv) \cdot D(h(v)\phi) = \int_\Omega f h(v)\phi$$

for all $h \in C_c^1(\mathbb{R})$, $\phi \in \mathscr{D}(\Omega)$. Indeed, using the fact that if v is a renormalized solution of (E)(f), then $\tilde{v}(t) \equiv v$ is a renormalized solution of (EP)$(\tilde{v}_0, \tilde{f})$ with $\tilde{v}_0 = v$, $\tilde{f}(t) \equiv f - b(v)$, the corresponding comparison and uniqueness result for the stationary problem is an immediate consequence of Theorem 2.3.

Remark 2.5. In general there is no uniqueness of the renormalized solution v itself. Trivially v is unique if b is strictly monotone. There is also uniqueness of v if, for example, a satisfies the Alt–Luckhaus assumptions and $b(r) = b(s) \Rightarrow a(r, \xi) = a(s, \xi)$ for all $r, s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$.

Remark 2.6. The result of Theorem 2.3 extends former results on uniqueness of weak, respectively, renormalized solutions contained in the literature. In [1] (cf. also, e.g., [7, 19, 21]) uniqueness results have only been obtained under the assumption of Lipschitz or Hölder continuity of a in the first variable. In a series of papers (cf. [9–11]) the first author has developed a method based on Kruzhkov's technique of doubling variables and proved uniqueness of weak solutions for semilinear diffusion–convection problems with continuous or discontinuous convection. His method has been adapted by other authors (cf., e.g., [15, 16]) to prove uniqueness results for elliptic-parabolic problems in some particular cases. The proof of our uniqueness theorem is also based on this method which is fully developed in [11].

3. PROOF OF THEOREM

A first step in the proof is

LEMMA 3.1. *Let v be a renormalized solution of* (EP)(v_0, f). *Then*

$$\begin{aligned}
&\int_Q \zeta_t \operatorname{sign}_0^+(v-k) \int_k^v h(r)\,db(r) + \int_\Omega \zeta(0,\cdot) \operatorname{sign}_0^+(v_0-k) \int_k^{v_0} h(r)\,db(r)\\
&\qquad + \int_Q \operatorname{sign}_0^+(v-k)\,\zeta f h(v)\\
&\qquad \geqslant \int_Q \operatorname{sign}_0^+(v-k)[(h(v)\,a(v,Dv) - h(k)\,a(k,0))\\
&\qquad\qquad \cdot D\zeta + \zeta h'(v)\,a(v,Dv)\cdot Dv],
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
&\int_Q \zeta_t \operatorname{sign}_0^+(-k-v) \int_{-k}^v h(r)\,db(r) + \int_\Omega \zeta(0,\cdot) \operatorname{sign}_0^+(-k-v_0) \int_{-k}^{v_0} h(r)\,db(r)\\
&\qquad + \int_Q \operatorname{sign}_0^+(-k-v)\,\zeta f h(v)\\
&\qquad \leqslant \int_Q \operatorname{sign}_0^+(-k-v)[(h(v)\,a(v,Dv) - h(-k)\,a(-k,0))\\
&\qquad\qquad \cdot D\zeta + \zeta h'(v)\,a(v,Dv)\cdot Dv],
\end{aligned} \tag{11}$$

for any $h \in C_c^1(\mathbb{R})$, $h \geqslant 0$, and any pair (k, ζ) satisfying

$$(k \in \mathbb{R}, \zeta \in \mathscr{D}([0,T[\times\Omega), \zeta \geqslant 0) \qquad \textit{or} \qquad (k \in \mathbb{R}^+, \zeta \in \mathscr{D}([0,T[\times\mathbb{R}^N, \zeta \geqslant 0). \tag{12}$$

Proof. Note that if v is a renormalized solution of (EP)(v_0, f), then $-v$ is a renormalized solution of the elliptic–parabolic problem associated with the equation $\tilde{b}(w)_t = \operatorname{div} \tilde{a}(w, Dw) + \tilde{f}$ where $\tilde{b}(r) = -b(-r)$, $\tilde{a}(r,\xi) = -a(-r,-\xi)$, $\tilde{f} = -f$ and initial data $\tilde{v}_0 = -v_0$. Therefore (10) implies (11) and we only give the proof of (10). Let $h \in C_c^1(\mathbb{R})$, $h \geqslant 0$. For $\varepsilon > 0$ let $H_\varepsilon \in W^{1,\infty}(\mathbb{R})$ be defined by $H_\varepsilon(r) = H(r/\varepsilon)$ where $H \in W^{1,\infty}(\mathbb{R})$ with $H(r) = 0$ for $r \leqslant 0$, $H(r) = r$ for $0 < r < 1$, $H(r) = 1$ if $r \geqslant 1$. Note that, for $\varepsilon > 0$, for any pair (k, ζ) satisfying (12), $H_\varepsilon(v-k)\zeta \in L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$. As v is a renormalized solution, $B_h(v) = \int_0^v h(r)\,db(r)$ ($\equiv B_h(T_l(v))$ for l sufficiently large) is in $L^1(Q)$, $B_h(v)_t \in L^{p'}(0,T; W^{-1,p'}(\Omega)) + L^1(Q)$

and $B_h(v)(0,\cdot)=\int_0^{v_0} h(r)\,db(r)$ in $W^{-1,p'}(\Omega)+L^1(\Omega)$. Applying Lemma 1.4 with $b(r)=B_h(r)=\int_0^r h(s)\,db(s)$, we find

$$\begin{aligned} -\int_Q fh(v)\,H_\varepsilon(v-k)\zeta+\int_Q a(v,Dv)\cdot D[h(v)\,H_\varepsilon(v-k)\zeta] &= -\int_0^T \langle B_h(v)_t, H_\varepsilon(v-k)\zeta\rangle\,dt \\ &= \int_Q \zeta_t \int_{v_0}^v H_\varepsilon(r-k)\,dB_h(r) \\ &= \int_Q \zeta_t \int_{v_0}^v H_\varepsilon(r-k)\,h(r)\,db(r). \end{aligned}$$

Passing to the limit with $\varepsilon\to 0$ on the right-hand side yields $\int_Q \zeta_t \times \int_{v_0}^v \operatorname{sign}_0^+(r-k)\,h(r)\,db(r) = \int_Q \zeta_t \operatorname{sign}_0^+(v-k)\int_k^v h(r)\,db(r) + \int_\Omega \zeta(0,\cdot)\times \operatorname{sign}_0^+(v_0-k)\int_k^{v_0} h(r)\,db(r)$. Moreover, $\lim_{\varepsilon\to 0}\int_Q fh(v)\,H_\varepsilon(v-k)\zeta = \int_Q \operatorname{sign}_0^+(v-k)\,fh(v)\zeta$. As to the second integral we have

$$\begin{aligned} \int_Q a(v,Dv)\cdot D[h(v)\,H_\varepsilon(v-k)\zeta] &= \int_Q H_\varepsilon(v-k)(h(v)\,a(v,Dv)\cdot D\zeta+\zeta h'(v)\,a(v,Dv)\cdot Dv) \\ &\quad +\frac{1}{\varepsilon}\int_{Q\cap\{0<v-k<\varepsilon\}} \zeta h(v)\,a(v,Dv)\cdot Dv =: I_1+I_2. \end{aligned}$$

It is clear that $\lim_{\varepsilon\to 0} I_1=\int \operatorname{sign}_0^+(v-k)(h(v)\,a(v,Dv)\cdot D\zeta+\zeta h'(v)\,a(v,Dv)\cdot Dv)$. Moreover, by monotonicity of a,

$$I_2 \geqslant \frac{1}{\varepsilon}\int_{\{0<v-k<\varepsilon\}} \zeta h(v)\,a(v,0)\cdot Dv =: I_2'.$$

As (k,ζ) satisfies (12), by the divergence theorem, we have

$$\begin{aligned} 0 &= \int_Q \operatorname{div}\left(\zeta\int_0^{\inf((v-k)^+/\varepsilon,\,1)} h(\varepsilon r+k)\,a(\varepsilon r+k,0)\right) \\ &= I_2' + \int_Q \int_0^{\inf((v-k)^+/\varepsilon,\,1)} h(\varepsilon r+k)\,a(\varepsilon r+k,0)\cdot D\zeta \end{aligned}$$

and thus $\liminf_{\varepsilon\to 0} I_2 \geqslant -\int_Q \operatorname{sign}_0^+(v-k)\,h(k)\,a(k,0)\cdot D\zeta$. Combining the preceding estimates we obtain the result. ∎

Using the preceding lemma we first prove the following local "renormalized Kato inequality":

PROPOSITION 3.2. *For $i=1, 2$, let $v_{0_i}\colon \Omega \to \overline{\mathbb{R}}$ be measurable with $b(v_{0_i}) \in L^1(\Omega)$, $f_i \in L^1(Q)$, v_i be a renormalized solution of* (EP)(v_{0_i}, f_i). *Then there exists $\kappa \in \operatorname{sign}^+(v_1 - v_2)$ such that, for a.e. $0<t<T$,*

$$-\int_{Q \cap \{v_1 > v_2\}} \zeta_t \int_{v_2}^{v_1} h(r)\, db(r) - \int_{\Omega \cap \{v_{0_1} > v_{0_2}\}} \zeta(0, \cdot) \int_{v_{0_2}}^{v_{0_1}} h(r)\, db(r)$$

$$+\int_{Q \cap \{v_1 > v_2\}} (h(v_1)\, a(v_1, Dv_1) - h(v_2)\, a(v_2, Dv_2)) \cdot D\zeta$$

$$+\int_{Q \cap \{v_1 > v_2\}} \zeta(h'(v_1)\, a(v_1, Dv_1) \cdot Dv_1 - h'(v_2)\, a(v_2, Dv_2) \cdot Dv_2)$$

$$\leqslant \int_Q \zeta\kappa(h(v_1)\, f_1 - h(v_2)\, f_2) \tag{13}$$

for any nonnegative $h \in C_c^1(\mathbb{R})$ and all nonnegative $\zeta \in \mathcal{D}([0, T[\times \Omega)$.

Proof. We choose two different pairs of variables (t, x), (s, y) and consider v_1, f_1 as functions in (s, y), v_2, f_2 in (t, x). Let $\zeta \in \mathcal{D}([0, T[\times \Omega)$, $\zeta \geqslant 0$, ρ_n a classical sequence of mollifiers in $\mathbb{R}^N$ and ϱ_l a sequence of mollifiers in $\mathbb{R}$ with $\operatorname{supp} \varrho_l \subset]-2/l, 0[$. Define

$$\zeta_{l,n}(t, x, s, y) = \zeta(t, x)\ \rho_n(x-y)\ \varrho_l(t-s).$$

Note that for l, n sufficiently large,

$$(s, y) \mapsto \zeta_{l,n}(t, x, s, y) \in \mathcal{D}([0, T[\times \Omega) \qquad \forall (t, x) \in Q \tag{14}$$

$$(t, x) \mapsto \zeta_{l,n}(t, x, s, y) \in \mathcal{D}([0, T[\times \Omega) \qquad \forall (s, y) \in Q. \tag{15}$$

Let $h \in C_c^1(\mathbb{R})$, $h \geqslant 0$, $H_\varepsilon \in W^{1,\infty}(\mathbb{R})$ defined as above. As v_1, v_2 are renormalized solutions, according to (3), for a.e. $(t, x) \in Q$,

$$\int_Q (\zeta_{l,n})_s \int_{v_{0_1}}^{v_1} h(r)\, H_\varepsilon(r - v_2(t, x))\, db(r) + \zeta_{l,n} f_1 h(v_1)\, H_\varepsilon(v_1 - v_2(t, x))\, dy\, ds$$

$$= \int_Q a(v_1, D_y v_1) \cdot D_y(h(v_1)\, H_\varepsilon(v_1 - v_2(t, x))\, \zeta_{l,n}\, dy\, ds$$

and, for a.e. $(s, y) \in Q$,

$$\int_Q (\zeta_{l,n})_t \int_{v_{0_2}}^{v_2} h(r)\, H_\varepsilon(v_1(s, y) - r)\, db(r) + \zeta_{l,n} f_2 h(v_2)\, H_\varepsilon(v_1(s, y) - r)\, dx\, dt$$

$$= \int_Q a(v_2, D_x v_2) \cdot D_x(h(v_2)\, H_\varepsilon(v_1(s, y) - v_2)\, \zeta_{l,n}\, dx\, dt.$$

Integrating both equations in (t, x), respectively, (s, y) over Q and taking their difference yields

$$\int_{Q\times Q}\left[(\zeta_{l,n})_s\int_{v_{0_1}}^{v_1} h(r)\,H_\varepsilon(r-v_2)\,db(r)-(\zeta_{l,n})_t\int_{v_{0_2}}^{v_2} h(r)\,H_\varepsilon(v_1-r)\,db(r)\right]$$
$$+\int_{Q\times Q}\zeta_{l,n}H_\varepsilon(v_1-v_2)(h(v_1)\,f_1-h(v_2)\,f_2)$$
$$=\int_{Q\times Q}(h(v_1)\,a(v_1,Dv_1)-h(v_2)\,a(v_2,Dv_2))\cdot D_{x+y}(\zeta_{l,n}H_\varepsilon(v_1-v_2))$$
$$+\int_{Q\times Q}(h'(v_1)\,a(v_1,Dv_1)\cdot D_y v_1-h'(v_2)\,a(v_2,Dv_2)\cdot D_x v_2)$$
$$\times H_\varepsilon(v_1-v_2)\,\zeta_{l,n}. \tag{16}$$

Denote the two integrals on the left by I_1, I_2, the two integrals on the right by I_3, I_4. In I_4, passing to the limit successively with $\varepsilon\to 0$, $l\to\infty$ and $n\to\infty$ yields

$$\lim_{\varepsilon,l,n} I_4=\int_{Q\cap\{v_1>v_2\}}\zeta(h'(v_1)\,a(v_1,Dv_1)\cdot Dv_1-h'(v_2)\,a(v_2,Dv_2)\cdot Dv_2). \tag{17}$$

As to I_2, recall that, for any $w, g\in L^1(Q)$, $\int_Q \alpha g\leqslant \inf_{\lambda>0} 1/\lambda\int_Q[(w+\lambda g)^+ - w^+]=\int_{\{w>0\}} g+\int_{\{w=0\}} g^+$ for all $\alpha\in \operatorname{sign}^+(w)$. Therefore

$$\lim_{\varepsilon\to 0} I_2=\int_{Q\times Q}\zeta_{l,n}\operatorname{sign}_0^+(v_1-v_2)(h(v_1)\,f_1-h(v_2)\,f_2)$$
$$\leqslant 1/\lambda\int_{Q\times Q}\zeta_{l,n}[(v_1-v_2+\lambda(h(v_1)\,f_1-h(v_2)\,f_2))^+-(v_1-v_2)^+]$$

for all $\lambda>0$, hence $\limsup_{\varepsilon,l,n} I_2\leqslant(1/\lambda)\int_Q\zeta[(v_1-v_2+\lambda(h(v_1)\,f_1-h(v_2)\,f_2))^+-(v_1-v_2)^+]$ for all $\lambda>0$. Consequently, we find

$$\limsup_{\varepsilon,l,n} I_2\leqslant\int_Q\zeta\kappa_h(h(v_1)\,f_1-h(v_2)\,f_2)$$
$$\leqslant\int_Q\zeta\kappa(h(v_1)\,f_1-h(v_2)\,f_2) \tag{18}$$

with $\kappa_h=\chi_{\{v_1>v_2\}}+\operatorname{sign}_0^+(h(v_1)\,f_1-h(v_2)\,f_2)\,\chi_{\{v_1=v_2\}}$ and $\kappa=\chi_{\{v_1>v_2\}}+\operatorname{sign}_0^+(f_1-f_2)\,\chi_{\{v_1=v_2\}}$. As to I_1, recall that $\operatorname{supp}\varrho_l\subset[-2/l, 0]$, hence

$$\lim_{\varepsilon\to 0} I_1 = \int_{(Q\times Q)\cap\{v_1>v_2\}} \left[(\zeta_{l,n})_s \int_{v_2}^{v_1} h(r)\,db(r) + (\zeta_{l,n})_t \int_{v_2}^{v_1} h(r)\,db(r) \right]$$

$$+\int_{Q\times\{0\}\times\Omega} \zeta_{l,n}(t,x,0,y)\,\mathrm{sign}_0^+(v_{0_1}-v_2) \int_{v_2}^{v_{0_1}} h(r)\,db(r)$$

$$+\int_{\{0\}\times\Omega\times Q} \zeta_{l,n}(0,x,s,y)\,\mathrm{sign}_0^+(v_1-v_{0_2}) \int_{v_{0_2}}^{v_1} h(r)\,db(r)$$

$$=\int_{(Q\times Q)\cap\{v_1>v_2\}} \varrho_l\rho_n\zeta_t(t,x) \int_{v_2}^{v_1} h(r)\,db(r)$$

$$+\int_{\{0\}\times\Omega\times Q} \zeta(0,x)\,\varrho_l(-s)\,\rho_n(x-y)\,\mathrm{sign}_0^+(v_1-v_{0_2}) \int_{v_{0_2}}^{v_1} h(r)\,db(r)$$

$$= I_{1,1}+I_{1,2}.$$

We have

$$\lim_{\varepsilon\to 0,\, l,n\to\infty} I_{1,1} = \int_{Q\cap\{v_1>v_2\}} \zeta_t \int_{v_2}^{v_1} h(r)\,db(r). \tag{19}$$

Consider the function

$$\phi_{l,n}(x,s,y) = \int_s^T \varrho_l(-r)\,dr\,\rho_n(x-y)\,\zeta(0,x)$$

$$= \int_{\inf(s,2/l)}^{2/l} \varrho_l(-r)\,dr\,\rho_n(x-y)\,\zeta(0,x).$$

Note that, for n sufficiently large, $\phi_{l,n}(x,\cdot)\in\mathscr{D}([0,T[\times\Omega)$, for any $x\in\Omega$. Applying Lemma 3.1 with $v=v_1$, $\zeta=\phi_{l,n}(x,\cdot)$ and $k=v_{0_2}$, we find

$$I_{1,2} = -\int_{\Omega\times[0,2/l]\times\Omega} (\phi_{l,n})_s\,\mathrm{sign}_0^+(v_1-v_{0_2}) \int_{v_{0_2}}^{v_1} h(r)\,db(r)$$

$$\leqslant \int_{\Omega\times\{0\}\times\Omega} \phi_{l,n}(x,0,y)\,\mathrm{sign}_0^+(v_{0_1}-v_{0_2}) \int_{v_{0_2}}^{v_{0_1}} h(r)\,db(r)$$

$$+\int_{\Omega\times[0,2/l]\times\Omega} \mathrm{sign}_0^+(v_1-v_{0_2})[h(v_1)\,f_1\phi_{l,n} - (h(v_1)\,a(v_1,Dv_1)$$

$$-h(v_{0_2})\,a(v_{0_2},0))\cdot D_y\phi_{l,n} - h'(v_1)\,a(v_1,Dv_1)\cdot Dv_1\phi_{l,n}].$$

It is clear that the second integral on the right converges to 0 as $l\to\infty$. Moreover, $\phi_{l,n}(x,0,y)=\int_0^T \varrho_l(-r)\,dr\,\rho_n(x-y)\,\zeta(0,x)=\rho_n(x-y)\,\zeta(0,x)$, and hence the first integral on the right converges to $\int_\Omega \zeta(0,x)\times \operatorname{sign}_0^+(v_{0_1}-v_{0_2})\int_{v_{0_2}}^{v_{0_1}} h(r)\,db(r)$ as $n\to\infty$. As a consequence,

$$\limsup_{\varepsilon,l,n} I_{1,2}\leqslant \int_\Omega \zeta(0,x)\operatorname{sign}_0^+(v_{0_1}-v_{0_2})\int_{v_{0_2}}^{v_{0_1}} h(r)\,db(r). \tag{20}$$

It remains to consider I_3. We have

$$\begin{aligned} I_3 &= \int_{Q\times Q} \varrho_l\,\rho_n H_\varepsilon(v_1-v_2)(h(v_1)\,a(v_1,Dv_1)-h(v_2)\,a(v_2,Dv_2))\cdot D_x\zeta(t,x)\\ &\quad+\frac{1}{\varepsilon}\int_{\{0<v_1-v_2<\varepsilon\}} \zeta_{l,n}(h(v_1)\,a(v_1,Dv_1)-h(v_2)\,a(v_2,Dv_2))\cdot(D_yv_1-D_xv_2)\\ &= I_{3,1}+I_{3,2}. \end{aligned}$$

Obviously

$$\lim_{\varepsilon\to 0,\,l,\,n\to\infty} I_{3,1}=\int_{Q\cap\{v_1>v_2\}} (h(v_1)\,a(v_1,Dv_1)-h(v_2)\,a(v_2,Dv_2))\cdot D\zeta(t,x). \tag{21}$$

In view of (16)–(21) it is now sufficient to show that $\liminf_{\varepsilon\to 0,\,l,\,n\to\infty} I_{3,2}\geqslant 0$. To this end note that

$$\begin{aligned} I_{3,2} &= \frac{1}{\varepsilon}\int_{(Q\times Q)\cap\{0<v_1-v_2<\varepsilon\}} \zeta_{l,n}(h(v_1)-h(v_2))\,a(v_1,Dv_1)\cdot(Dv_1-Dv_2)\\ &\quad+\frac{1}{\varepsilon}\int_{(Q\times Q)\cap\{0<v_1-v_2<\varepsilon\}} \zeta_{l,n}h(v_2)(a(v_1,Dv_1)-a(v_2,Dv_2))\\ &\quad\cdot(Dv_1-Dv_2), \end{aligned}$$

where both integrals on the right are well-defined: indeed, as v_1, v_2 are ε-close on the integration set, cutting of one of these two functions implies truncation of the other.

In the following, let $K>0$ be such that $\operatorname{supp} h\subset(-K,K)$. Then, for ε sufficiently small,

$$\begin{aligned} &\frac{1}{\varepsilon}\int_{(Q\times Q)\cap\{0<v_1-v_2<\varepsilon\}} |\zeta_{l,n}(h(v_1)-h(v_2))\,a(v_1,Dv_1)\cdot(Dv_1-Dv_2)|\\ &\qquad\leqslant \int_{(Q\times Q)\cap\{0<v_1-v_2<\varepsilon\}} \zeta_{l,n}L_hC(K)\\ &\qquad\quad\times(1+|DT_K(v_1)|^{p-1})(|DT_K(v_1)|+|DT_K(v_2)|), \end{aligned}$$

where $L_h = \sup h'$. Note that, for fixed l, n, the integrand of the last integral belongs to $L^1(Q\times Q)$ and thus the integral converges to 0 as $\varepsilon \to 0$. As to the remaining part, due to assumption (8), we have

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{(Q\times Q)\cap\{0<v_1-v_2<\varepsilon\}} \zeta_{l,n} h(v_2)(a(v_1, Dv_1)-a(v_2, Dv_2))\cdot(Dv_1-Dv_2)\\
&\quad \geqslant -\frac{1}{\varepsilon}\int_{\{0<v_1-v_2<\varepsilon\}} \zeta_{l,n} h(v_2)\, C(T_K(v_1), T_K(v_2))\\
&\qquad \times(1+|DT_K(v_1)|^p+|DT_K(v_2)|^p)\,|v_1-v_2|\\
&\qquad +\frac{1}{\varepsilon}\int_{\{0<v_1-v_2<\varepsilon\}} \zeta_{l,n} h(v_2)\, \Gamma(T_K(v_1), T_K(v_2))\cdot DT_K(v_1)\\
&\qquad +\frac{1}{\varepsilon}\int_{\{0<v_1-v_2<\varepsilon\}} \zeta_{l,n} h(v_2)\, \hat{\Gamma}(T_K(v_1), T_K(v_2))\cdot DT_K(v_2)\\
&\quad = J_1+J_2+J_3.
\end{aligned}$$

Using the same arguments as above we obtain $\lim_{\varepsilon\to 0} J_1 = 0$. Moreover, according to the divergence theorem, we have

$$\begin{aligned}
J_2 = -\int_{Q\times Q} & h(v_2)\\
&\times\int_0^{\inf(((T_K(v_1(s,y))-T_K(v_2)))^+)/\varepsilon,\,1)} \Gamma(\varepsilon r+T_K(v_2), T_K(v_2))\,dr\,D_y\zeta_{l,n}
\end{aligned}$$

and

$$J_3 = \int_{Q\times Q} h(v_1)\int_0^{\inf(((T_K(v_1)-T_K(v_2)(t,x))^+)/\varepsilon,\,1)} \hat{\Gamma}(T_K(v_1), T_K(v_1)-\varepsilon r)\,dr\,D_x\zeta_{l,n}.$$

Due to the continuity of Γ, $\hat{\Gamma}$, it follows that

$$\begin{aligned}
\liminf_{\varepsilon\to 0} I_{3,2} \geqslant \int_{\{0<v_1-v_2<\varepsilon\}} & \hat{\Gamma}(T_K(v_1), T_K(v_1))\cdot D_x\zeta_{l,n}\\
&-\Gamma(T_K(v_2), T_K(v_2))\cdot D_y\zeta_{l,n} = 0
\end{aligned}$$

as $\Gamma(r, r) = \hat{\Gamma}(r, r) = 0$ for all $r\in\mathbb{R}$ and the assertion follows. ∎

Now we are in the position to give the

Proof of Theorem 2.3. The main step in the proof is to show that the local estimate of Proposition 3.2 holds globally, i.e., (13) still holds for $\zeta \in \mathscr{D}([0, T[\times\mathbb{R}^N)$. Indeed, assuming this for the moment, let v_1, v_2 be renormalized solutions of $(\mathrm{EP})(v_{0_1}, f_1)$, $(\mathrm{EP})(v_{0_2}, f_2)$, respectively. Choosing $\zeta = \alpha \otimes 1$ with $\alpha \in \mathscr{D}([0, T[)$ in (13) yields

$$-\int_Q \alpha_t \left[\operatorname{sign}_0^+(v_1 - v_2) \int_{v_2}^{v_1} h(r)\, db(r) - \operatorname{sign}_0^+(v_{0_1} - v_{0_2}) \int_{v_{0_2}}^{v_{0_1}} h(r)\, db(r) \right]$$

$$+ \int_{\{v_1 > v_2\}} \alpha(h'(v_1)\, a(v_1, Dv_1) \cdot Dv_1 - h'(v_2)\, a(v_2, Dv_2)) \cdot Dv_2)$$

$$\leqslant \int_Q \alpha\kappa(h(v_1)\, f_1 - h(v_2)\, f_2) \tag{22}$$

for any $h \in W^{1,\infty}(\mathbb{R})$ with compact support, for some $\kappa \in \operatorname{sign}^+(v_1 - v_2)$. Let $h = h_n \in W^{1,\infty}(\mathbb{R})$ be defined by $h_n(r) = \inf((n+1-|r|)^+, 1)$ and pass to the limit in the inequality with $n \to \infty$. As to the second integral, we have

$$\int_{\{v_1 > v_2\}} \alpha(h'(v_1)\, a(v_1, Dv_1) \cdot Dv_1 - h'(v_2)\, a(v_2, Dv_2)) \cdot Dv_2)$$

$$= \int_{\{v_1 > v_2\}} \alpha h'(v_1)(a(v_1, Dv_1) - a(v_1, 0)) \cdot Dv_1$$

$$- \int_{\{v_1 > v_2\}} \alpha h'(v_2)(a(v_2, Dv_2) - a(v_2, 0)) \cdot Dv_2$$

$$+ \int_{\{v_1 > v_2\}} \alpha(h'(v_1)\, a(v_1, 0) \cdot Dv_1 + h'(v_2)\, a(v_2, 0) \cdot Dv_2).$$

As v_1, v_2 are renormalized solutions, according to (4), the first two integrals on the right tend to 0 as $n \to \infty$. Moreover, by the divergence theorem,

$$\int_{\{v_1 > v_2\}} \alpha(h'(v_1)\, a(v_1, 0) \cdot Dv_1 + h'(v_2)\, a(v_2, 0) \cdot Dv_2)$$

$$= \int_Q \alpha \operatorname{div} \left(\int_{\inf(v_1, v_2)}^{v_1} h'(r)\, a(r, 0)\, dr \right) = 0;$$

hence, in (22), the second integral converges to 0 as $h = h_n \to 1$. Consequently, in the limit, we obtain

$$-\int_Q \alpha_t[(b(v_1) - b(v_2))^+ - (b(v_{0_1}) - b(v_{0_2}))^+] \leqslant \int_Q \alpha\kappa(f_1 - f_2)$$

for all $\alpha \in \mathcal{D}([0, T[)$ and the proof is complete. Let us now prove that (13) holds for any $\zeta \in \mathcal{D}([0, T[\times \mathbb{R}^N)$. Using a partition of unity subordinate to a covering of $\bar{\Omega}$ by balls B_i, $i = 0, ..., n$ satisfying $B_0 \cap \partial\Omega = \emptyset$ and, for $i \neq 0$, $B_i \subset\subset B_i'$ with $B_i' \cap \partial\Omega$ is part of the graph of a Lipschitz function, we may assume that $\operatorname{supp} \zeta \subset B = B_i$ for some $i \neq 0$. Again we choose two pairs of variables (s, y), (t, x), consider v_1, f_1 as functions in (s, y), v_2, f_2 in (t, x) and choose ϱ_l mollifiers in $\mathbb{R}$ with $\operatorname{supp} \varrho_l \subset \in]-2/l, 0[$. As $B \cap \partial\Omega$ is part of the graph of a Lipschitz function, it is easy to see that there exists a sequence of mollifiers ρ_n in $\mathbb{R}^N$ such that $x \mapsto \rho_n(x-y) \in \mathcal{D}(\Omega)$ for all $y \in B$, $\sigma_n(x) = \int_\Omega \rho_n(x-y)\, dy$ is an increasing sequence for all $x \in B$ and $\sigma_n(x) = 1$ for any $x \in B$ with $\mathrm{d}(x, \mathbb{R}^N \backslash \Omega) > c/n$ (with $c = C(i)$ depending on $B = B_i$). Define

$$\zeta_{l,n}(t, x, s, y) = \zeta(t, x)\, \rho_n(x-y)\, \varrho_l(t-s).$$

Note that, for l, n sufficiently large,

$$\begin{aligned} &(s, y) \mapsto \zeta_{l,n}(t, x, s, y) \in \mathcal{D}(]0, T[\times \bar{\Omega} \qquad \text{for all} \quad (t, x) \in Q, \\ &(t, x) \mapsto \zeta_{l,n}(t, x, s, y) \in \mathcal{D}([0, T[\times \Omega \qquad \text{for all} \quad (s, y) \in Q, \end{aligned}$$

and the function

$$\begin{aligned} \hat{\zeta}_n &= \int_Q \zeta_{l,n}(t, x, s, y)\, dy\, ds \\ &= \zeta(t, x) \int_\Omega \rho_n(x-y)\, dy \int_0^T \varrho_l(t-s)\, ds = \zeta\sigma_n \end{aligned} \tag{23}$$

satisfies

$$\hat{\zeta}_n \in \mathcal{D}([0, T[\times \Omega), \qquad 0 \leqslant \hat{\zeta}_m \leqslant \hat{\zeta}_n \leqslant \zeta \qquad \text{for any} \quad m \leqslant n. \tag{24}$$

Applying Lemma 3.1 with $v = v_1$, $k = 0$, $\zeta = \zeta_{l,n}$, and $h(\cdot)\, H_\varepsilon(\cdot - v_2^+)$ in the place of h yields

$$\begin{aligned} &\int_{Q \times Q} (\zeta_{l,n})_s \int_{v_2^+}^{v_1^+} h(r)\, H_\varepsilon(r - v_2^+)\, db(r) \\ &\quad + \int_{Q \times \{0\} \times \Omega} \zeta_{l,n}(t, x, 0, y) \int_{v_2^+}^{v\,0_1^+} h(r)\, H_\varepsilon(r - v_2^+)\, db(r) \\ &\quad + \int_{Q \times Q} f_1 h(v_1)\, H_\varepsilon(v_1^+ - v_2^+)\, \zeta_{l,n} \\ &\geqslant \int_{Q \times Q} a(v_1, Dv_1) \cdot D_y[h(v_1)\, H_\varepsilon(v_1^+ - v_2^+)\, \zeta_{l,n}]. \end{aligned}$$

Moreover, as v_2 is a renormalized solution of (EP)(v_{0_2}, f), according to (3), we have

$$\int_{Q\times Q} (\zeta_{l,n})_t \int_{v_1^+}^{v_2} h(r)\, H_\varepsilon(v_1^+ - r^+)\, db(r)$$

$$+\int_{\{0\}\times\Omega\times Q} \zeta_{l,n}(0,x,s,y) \int_{v_1^+}^{v_{0_2}} h(r)\, H_\varepsilon(v_1^+ - r^+)\, db(r)$$

$$+\int_{Q\times Q} f_2 h(v_2)\, H_\varepsilon(v_1^+ - v_2^+)\, \zeta_{l,n}$$

$$=\int_{Q\times Q} a(v_2, Dv_2)\cdot D_x[h(v_2)\, H_\varepsilon(v_1^+ - v_2^+)\, \zeta_{l,n}].$$

Denote $I_1^{(1)}, ..., I_4^{(1)}$, respectively $I_1^{(2)}, ...I_4^{(2)}$, the integrals arising in the preceding two estimates and take the difference of both. As in the local estimate our aim is to pass to the limit successively in $\varepsilon \to 0$, $l \to \infty$ and $n \to \infty$. We have

$$\lim_{\varepsilon\to 0} I_1^{(2)} = \int_{Q\times Q} (\zeta_{l,n})_t \operatorname{sign}_0^+(v_1^+ - v_2^+) \int_{v_1^+}^{v_2} h(r)\, db(r)$$

$$= \int_{Q\times Q} (\zeta_{l,n})_t \operatorname{sign}_0^+(v_1^+ - v_2^+) \int_{v_1^+}^{v_2^+} h(r)\, db(r)$$

$$+\int_{\{v_1>0\}\cap\{v_2<0\}} (\zeta_{l,n})_t \int_0^{v_2} h(r)\, db(r),$$

hence

$$I_1 := \lim_{\varepsilon\to 0} [I_1^{(1)} - I_1^{(2)}] = \int_{Q\times Q} \zeta_t(t,x)\, \varrho_l\, \rho_n \operatorname{sign}_0^+(v_1^+ - v_2^+) \int_{v_2^+}^{v_1^+} h(r)\, db(r)$$

$$-\int_{\{v_1>0\}\cap\{v_2<0\}} (\zeta_{l,n})_t \int_0^{v_2} h(r)\, db(r).$$

Recall that supp $\varrho_l \subset\,]-2/l, 0[$, hence $I_2^{(1)} = 0$ and

$$I_2 := \lim_{\varepsilon\to 0} [I_2^{(1)} - I_2^{(2)}]$$

$$= \int_{\Omega\times\,]-2/l,\,0[\,\times\Omega} \zeta_{l,n}(0,x,s,y) \operatorname{sign}_0^+(v_1^+ - v_{0_2}^+) \int_{v_{0_2}}^{v_1^+} h(r)\, db(r)$$

$$= \int_{\Omega\times\,]-2/l,\,0[\,\times\Omega} \zeta_{l,n}(0,x,s,y) \operatorname{sign}_0^+(v_1^+ - v_{0_2}^+) \int_{v_{0_2}^+}^{v_1^+} h(r)\, db(r)$$

$$+\int_{\Omega\times\,]-2/l,\,0[\,\times\Omega} \zeta_{l,n}(0,x,s,y)\, \chi_{\{v_1>0\}\cap\{v_{0_2}<0\}} \int_{v_{0_2}}^{0} h(r)\, db(r).$$

As to the third term, we have

$$I_3 := \lim_{\varepsilon \to 0} [I_3^{(1)} - I_3^{(2)}] = \int_{Q \times Q \cap \{v_1^+ > v_2^+\}} (h(v_1) f_1 - h(v_2) f_2) \zeta_{l,n}.$$

Now consider the right-hand side. We have

$$\begin{aligned} I_4^{(1)} - I_4^{(2)} \\ &= \int_{Q \times Q} (h(v_1) a(v_1, Dv_1) - h(v_2) a(v_2^+, Dv_2^+)) \cdot D_{x+y}[H_\varepsilon(v_1 - v_2^+) \zeta_{l,n}] \\ &\quad + \int_{Q \times Q} \zeta_{l,n} H_\varepsilon(v_1 - v_2^+)(h'(v_1) a(v_1, Dv_1) \cdot D_y v_1 \\ &\quad - h'(v_2) a(v_2^+, Dv_2^+)) \cdot D_x v_2^+) \\ &\quad - \int_{(Q \times Q) \cap \{v_2 < 0\}} a(v_2, Dv_2) \cdot D_x[h(v_2) \zeta_{l,n}] H_\varepsilon(v_1) \end{aligned}$$

Using condition (8), we can prove exactly as in the proof of the local estimate that

$$\begin{aligned} \liminf_{\varepsilon \to 0} \int_{Q \times Q} (h(v_1) a(v_1, Dv_1) - h(v_2) a(v_2^+, Dv_2^+)) \cdot D_{x+y}[H_\varepsilon(v_1 - v_2^+) \zeta_{l,n}] \\ \geqslant \int_{(Q \times Q) \cap \{v_1 > v_2^+\}} \varrho_l \rho_n (h(v_1) a(v_1, Dv_1) - h(v_2) a(v_2^+, Dv_2^+)) \\ \cdot D_x \zeta(t, x) =: I_{4,1}. \end{aligned}$$

In the same way it is clear that

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_{Q \times Q} (h'(v_1) a(v_1, Dv_1) \cdot D_y v_1 - h'(v_2) a(v_2^+, Dv_2^+)) \\ \cdot D_x v_2^+) H_\varepsilon(v_1 - v_2^+) \zeta_{l,n} \\ = \int_{(Q \times Q) \cap \{v_1 > v_2^+\}} \zeta_{l,n} (h'(v_1) a(v_1, Dv_1) \cdot Dv_1 \\ - h'(v_2) a(v_2^+, Dv_2^+)) \cdot Dv_2^+) =: I_{4,2}. \end{aligned}$$

As to the remaining term, note that, by Lemma 3.1, $\int_{Q \cap \{v_2 < 0\}} (\zeta_{l,n})_t \times \int_{v_{0_2}}^{v_2} h(r)\, db(r) + \zeta_{l,n} h(v_2) f_2 - a(v_2, Dv_2) D[h(v_2) \zeta_{l,n}] \leqslant 0$, hence

$$
\begin{aligned}
&-\int_{(Q\times Q)\cap\{v_2<0\}} a(v_2, Dv_2)\cdot D_x[h(v_2)\,\zeta_{l,n}]\,H_\varepsilon(v_1)\\
&\quad\geqslant \int_{(Q\times Q)\cap\{v_2<0\}} (\zeta_{l,n})_t \int_{v_{0_2}}^{v_2} h(r)\,db(r)\\
&\qquad + \zeta_{l,n} h(v_2)\, f_2 - a(v_2, Dv_2)\cdot D_x[h(v_2)\,\zeta_{l,n}]\\
&\qquad - \int_{(Q\times Q)\cap\{v_2<0\}} H_\varepsilon(v_1)\,h(v_2)\,f_2\zeta_{l,n}\\
&\qquad - \int_{(Q\times Q)\cap\{v_2<0\}} (\zeta_{l,n})_t \int_{v_{0_2}}^{v_2} h(r)\,db(r)\,H_\varepsilon(v_1)\\
&\quad\to_{\varepsilon\to 0} \int_{(Q\times Q)\cap\{v_2<0\}} (\zeta_{l,n})_t \int_{v_{0_2}}^{v_2} h(r)\,db(r)\\
&\qquad + \zeta_{l,n} h(v_2)\, f_2 - a(v_2, Dv_2)\cdot D_x[h(v_2)\,\zeta_{l,n}]\\
&\qquad - \int_{(Q\times Q)\cap\{v_2<0\}\cap\{v_1>0\}} h(v_2)\,f_2\zeta_{l,n}\\
&\qquad - \int_{(Q\times Q)\cap\{v_2<0\}\cap\{v_1>0\}} (\zeta_{l,n})_t \int_{v_{0_2}}^{v_2} h(r)\,db(r)\\
&\quad =: I_{4,3} + I_{4,4} + I_{4,5}.
\end{aligned}
$$

Putting on the left-hand side of our inequality $I_{4,4}$ and $I_{4,5}$, we may now pass to the limit with $l, n\to\infty$. Consider first $J_1 := I_1 + I_2 + I_{4,5}$. We have

$$
\begin{aligned}
J_1 = &\int_{Q\times Q} \zeta_t(t,x)\,\varrho_l\,\rho_n\,\operatorname{sign}_0^+(v_1^+ - v_2^+) \int_{v_2^+}^{v_1^+} h(r)\,db(r)\\
&+ \int_{\Omega\times]-2/l,0[\times\Omega} \zeta(0,x)\,\varrho_l(-s)\,\rho_n(x-y)\,\operatorname{sign}_0^+(v_1^+ - v_{0_2}^+) \int_{v_{0_2}^+}^{v_1^+} h(r)\,db(r).
\end{aligned}
$$

Let us introduce the function

$$
\begin{aligned}
\phi_{l,n}(x,s,y) &= \zeta(0,x)\,\rho_n(x-y)\int_s^T \varrho_l(-r)\,dr\\
&= \zeta(0,x)\,\rho_n(x-y)\int_{\inf(s,2/l)}^{2/l} \varrho_l(-r)\,dr.
\end{aligned}
$$

Note that, for any $x\in\Omega$, $(s,y)\mapsto\phi_{l,n}(x,s,y)\in\mathscr{D}([0,T[\times\mathbb{R}^N)$, and as v_1 is a renormalized solution, according to Lemma 3.1, we have

$$\int_{\Omega\times]-2/l,0[\times\Omega} \zeta(0,x)\,\varrho_l(-s)\,\rho_n(x-y)\,\mathrm{sign}_0^+(v_1^+-v_{0_2}^+)\int_{v_{0_2}^+}^{v_1^+} h(r)\,db(r)$$

$$= -\int_{\Omega\times]-2l,0[\times\Omega} (\phi_{l,n})_s\,\mathrm{sign}_0^+(v_1^+-v_{0_2}^+)\int_{v_{0_2}^+}^{v_1^+} h(r)\,db(r)$$

$$\leqslant \int_{\Omega\times\Omega} \phi_{l,n}(x,0,y)\,\mathrm{sign}_0^+(v_{0_1}^+-v_{0_2}^+)\int_{v_{0_2}^+}^{v_{0_1}^+} h(r)\,db(r)$$

$$+\int_{\Omega\times]-2/l,0[\times\Omega} \mathrm{sign}_0^+(v_1-v_{0_2}^+)\,h(v_1)\,f_1\,\phi_{l,n}(x,s,y)$$

$$-\int_{\Omega\times]-2/l,0[\times\Omega} \mathrm{sign}_0^+(v_1-v_{0_2}^+)[(h(v_1)\,a(v_1,Dv_1)$$

$$-h(v_{0_2}^+)\,a(v_{0_2}^+,0))\cdot D_y\phi_{l,n}+h'(v_1)\,a(v_1,Dv_1)\cdot Dv_1\,\phi_{l,n}].$$

It is clear that the last two integrals on the right tend to 0 if $l\to\infty$. Moreover, note that $\phi_{l,n}(x,0,y)=\zeta(0,x)\,\rho_n(x-y)$ and thus the first integral on the right converges to $\int_\Omega \zeta(0,x)\,\mathrm{sign}_0^+(v_{0_1}^+-v_{0_2}^+)\int_{v_{0_2}^+}^{v_{0_1}^+} h(r)\,db(r)$ as $n\to\infty$. Consequently

$$\liminf_{l,n\to\infty} J_1 \leqslant \int_Q \zeta_t\,\mathrm{sign}_0^+(v_1^+-v_2^+)\int_{v_2^+}^{v_1^+} h(r)\,db(r)$$

$$+\int_\Omega \zeta(0,x)\,\mathrm{sign}_0^+(v_{0_1}^+-v_{0_2}^+)\int_{v_{0_2}^+}^{v_{0_1}^+} h(r)\,db(r).$$

Next consider

$$J_2 := I_3+I_{4,4} = \int_{\{v_1>v_2^+\}} \zeta_{l,n}[h(v_1^+)\,f_1-(1-\mathrm{sign}_0^+(-v_2))\,h(v_2)\,f_2]$$

$$= \int_{\{v_1>v_2^+\}} \zeta_{l,n}\,\mathrm{sign}_0^+(v_1)[h(v_1^+)\,f_1-(1-\mathrm{sign}_0^+(-v_2))\,h(v_2)\,f_2].$$

Argueing as in the proof of Proposition 3.3 we obtain

$$\limsup_{l,n\to\infty} J_2 \leqslant \int_Q \zeta\kappa_+\,\mathrm{sign}_0^+(v_1)(h(v_1)\,f_1-(1-\mathrm{sign}_0^+(-v_2))\,h(v_2)\,f_2)$$

with $\kappa_+\in\mathrm{sign}^+(v_1^+-v_2^+)$. It remains to consider the right-hand side. Obviously

$$\lim_{l,n\to\infty} I_{4,1}+I_{4,2} = \int_{Q\cap\{v_1>v_2^+\}} (h(v_1)\,a(v_1,Dv_1)-h(v_2)\,a(v_2^+,Dv_2^+))\cdot D\zeta$$

$$+\zeta(h'(v_1)\,a(v_1,Dv_1)\cdot Dv_1-h'(v_2)\,a(v_2^+,Dv_2^+))\cdot Dv_2^+)$$

as $l, n \to \infty$. Moreover, we have (see (23))

$$I_{4,3} = \int_{Q \cap \{v_2 < 0\}} (\hat{\zeta}_n)_t \int_{v_{0_2}}^{v_2} h(r)\, db(r) + \hat{\zeta}_n h(v_2)\, f_2 - a(v_2, Dv_2) \cdot D_x[h(v_2)\, \hat{\zeta}_n]\, dx\, dt.$$

By Lemma 3.1, the functional $\mathscr{L}\colon \mathscr{D}([0, T[\, \times \mathbb{R}^N) \to \mathbb{R}$ defined by

$$\zeta \mapsto \int_{Q \cap \{v_2 < 0\}} \zeta_t \int_{v_{0_2}}^{v_2} h(r)\, db(r) + \zeta h(v_2)\, f_2 - a(v_2, Dv_2)) \cdot D_x[h(v_2)\zeta]$$

is monotone decreasing. By (24) we have $\mathscr{L}(\zeta) \leqslant \mathscr{L}(\hat{\zeta}_n) \leqslant \mathscr{L}(\hat{\zeta}_m) \leqslant 0$ and thus $\mathscr{L}(\hat{\zeta}_n)(=I_{4,3})$ converges as $n \to \infty$. Combining the preceding estimates yields

$$\begin{aligned}
&\int_Q \zeta_t \operatorname{sign}_0^+(v_1^+ - v_2^+) \int_{v_2^+}^{v_1^+} h(r)\, db(r) \\
&\quad + \int_\Omega \zeta(0, x) \operatorname{sign}_0^+(v_{0_1}^+ - v_{0_2}^+) \int_{v_{0_2}^+}^{v_{0_1}^+} h(r)\, db(r) \\
&\quad + \int_Q \zeta\kappa_+ \operatorname{sign}_0^+(v_1)(h(v_1)\, f_1 - (1 - \operatorname{sign}_0^+(-v_2))\, h(v_2)\, f_2) \\
&\geqslant \int_{Q \cap \{v_1 > v_2^+\}} (h(v_1)\, a(v_1, Dv_1) - h(v_2)\, a(v_2^+, Dv_2^+)) \cdot D\zeta \\
&\quad + \zeta(h'(v_1)\, a(v_1, Dv_1) \cdot Dv_1 - h'(v_2)\, a(v_2^+, Dv_2^+)) \cdot Dv_2^+) \\
&\quad + \lim_{n \to \infty} \int_{Q \cap \{v_2 < 0\}} (\hat{\zeta}_n)_t \int_{v_{0_2}}^{v_2} h(r)\, db(r) \\
&\quad + h(v_2)\, f_2 \hat{\zeta}_n - a(v_2, Dv_2) \cdot D[h(v_2)\, \hat{\zeta}_n]\, dx\, dt.
\end{aligned}$$

Recall that, if v is a renormalized solution of (EP)(v_0, f), $-v$ is a renormalized solution of the elliptic-parabolic problem with b being replaced by $\tilde{b}(r) = -b(-r)$, a by $\tilde{a}(r, \xi) = -a(-r, -\xi)$ and data $-v_0, -f$. Thus changing v_1 to $-v_2$, v_2 to $-v_1$, v_{0_1} to $-v_{0_2}$, v_{0_2} to $-v_{0_1}$, f_1 to $-f_2$, f_2 to $-f_1$, using the same arguments as above, we obtain the existence of $\kappa_- \in \operatorname{sign}^+(v_2^- - v_1^-)$ such that

$$\int_Q \zeta_t \operatorname{sign}_0^+(v_2^- - v_1^-) \int_{-v_2^-}^{-v_1^-} h(r)\, db(r)$$

$$+\int_\Omega \zeta(0, x) \operatorname{sign}_0^+(v_{0_2}^- - v_{0_1}^-) \int_{-v_{0_2}^-}^{-v_{0_1}^-} h(r)\, db(r)$$

$$+\int_Q \zeta\kappa_- \operatorname{sign}_0^+(v_2^-)(h(v_1)(1-\operatorname{sign}_0^+(v_1^+))\, f_1 - h(v_2)\, f_2)$$

$$\geqslant \int_{Q\cap\{v_2^- > v_1^-\}} (h(v_1)\, a(-v_1^-, -Dv_1^-)) - h(v_2)\, a(v_2, Dv_2)\cdot D\zeta$$

$$+\zeta(h'(v_1)\, a(-v_1^-, -Dv_1^-)\cdot Dv_1 - h'(v_2)\, a(v_2, Dv_2))\cdot Dv_2$$

$$-\lim_{n\to\infty} \int_{Q\cap\{v_1>0\}} (\hat{\zeta}_n)_t \int_{v_{0_1}}^{v_1} h(r)\, db(r) + h(v_1)\, f_1 \hat{\zeta}_n - a(v_1, Dv_1)$$

$$\cdot D[h(v_1)\, \hat{\zeta}_n]\, dx\, dt.$$

As $\kappa = (1-\operatorname{sign}_0^+(v_1^+)) \operatorname{sign}_0^+(-v_2)\, \kappa_- + \operatorname{sign}_0^+(v_1^+)\, \kappa_+ = (1-\operatorname{sign}_0^+(-v_2)) \times \operatorname{sign}_0^+(v_1)\, \kappa_+ + \operatorname{sign}_0^+(-v_2)\, \kappa_- \in \operatorname{sign}^+(v_1 - v_2)$, taking the sum of the preceding two inequalities yields

$$\int_Q \zeta_t \operatorname{sign}_0^+(v_1 - v_2) \int_{v_2}^{v_1} h(r)\, db(r) + \int_\Omega \zeta(0, x) \operatorname{sign}_0^+(v_{0_1} - v_{0_2}) \int_{v_{0_2}}^{v_{0_1}} h(r)\, db(r)$$

$$+\int_Q \zeta\kappa(h(v_1)\, f_1 - h(v_2)\, f_2)$$

$$-\int_{Q\cap\{v_1>v_2\}} (h(v_1)\, a(v_1, Dv_1) - h(v_2)\, a(v_2, Dv_2))\cdot D\zeta$$

$$-\int_{Q\cap\{v_1>v_2\}} \zeta(h'(v_1)\, a(v_1, Dv_1)\cdot Dv_1 - h'(v_2)\, a(v_2, Dv_2))\cdot Dv_2)$$

$$\geqslant \lim_{n\to\infty} \int_{Q\cap\{v_2<0\}} (\hat{\zeta}_n)_t \int_{v_{0_2}}^{v_2} h(r)\, db(r) + h(v_2)\, f_2 \hat{\zeta}_n - a(v_2, Dv_2)$$

$$\cdot D[h(v_2)\, \hat{\zeta}_n]\, dx\, dt$$

$$-\lim_{n\to\infty} \int_{Q\cap\{v_1>0\}} (\hat{\zeta}_n)_t \int_{v_{0_1}}^{v_1} h(r)\, db(r) + h(v_1)\, f_1 \hat{\zeta}_n$$

$$-a(v_1, Dv_1)\cdot D[h(v_1)\, \hat{\zeta}_n]\, dx\, dt \tag{25}$$

(actually, as in the local estimate (13), we may always assume that $\kappa = \chi_{\{v_1>v_2\}} + \operatorname{sign}_0^+(f_1 - f_2)\, \chi_{\{v_1=v_2\}}$). Now let $\mathscr{F}: \mathscr{D}([0, T[\times B) \to \mathbb{R}$ be the functional defined by $\mathscr{F}(\zeta) =$ left-hand side of inequality (25). In terms of $\mathscr{F}$ what remains to be proved is the positivity of $\mathscr{F}$. To this end, note that, for any $\zeta \in \mathcal{D}([0, T[\times B)$, $\sigma_m \zeta \in \mathcal{D}([0, T[\times \Omega)$ for any $m \in \mathbb{N}$. According to the local estimate of Proposition 3.3 $\mathcal{F}(\sigma_m \zeta) \geqslant 0$ for all m, hence $\mathcal{F}(\zeta) = \mathcal{F}(\sigma_m \zeta) + \mathcal{F}((1-\sigma_m)\zeta) \geqslant \mathcal{F}((1-\sigma_m)\zeta)$ for all m. Consequently, it is sufficient to show that $\liminf_{m \to \infty} \mathcal{F}((1-\sigma_m)\zeta) \geqslant 0$. Note that, for n sufficiently large, $\sigma_n = 1$ on $\operatorname{supp} \sigma_m$, hence $(1-\sigma_m)\,\sigma_n = \sigma_n - \sigma_m$ and thus, by (25), we obtain (recall that $\hat{\zeta}_n = \sigma_n \zeta$)

$$\liminf_{m \to \infty} \mathcal{F}((1-\sigma_m)\zeta)$$

$$\geqslant \lim_{m \to \infty} \lim_{n \to \infty} \int_{Q \cap \{v_2 < 0\}} (\sigma_n - \sigma_m)\,\zeta_t \int_{v_{0_2}}^{v_2} h(r)\,db(r)$$

$$+ (\sigma_n - \sigma_m)\,\zeta h(v_2)\,f_2 - a(v_2, Dv_2) \cdot D[h(v_2)(\sigma_n - \sigma_m)\zeta]\,dx\,dt$$

$$- \lim_{m \to \infty} \lim_{n \to \infty} \int_{Q \cap \{v_1 > 0\}} (\sigma_n - \sigma_m)\,\zeta_t \int_{v_{0_1}}^{v_1} h(r)\,db(r) + (\sigma_n - \sigma_m)\,\zeta h(v_1)\,f_1$$

$$- a(v_1, Dv_1) \cdot D[h(v_1)(\sigma_n - \sigma_m)\zeta]\,dx\,dt = 0$$

and the proof is complete. ∎

4. EXTENSIONS AND REMARKS

Using the same arguments as above we can prove a more general result than Theorem 2.3.

DEFINITION 4.1. A measurable function $v \colon \Omega \to \mathbb{R}$ is a *renormalized subsolution* (respectively, *renormalized supersolution*) of $(\mathrm{EP})(v_0, f)$ if v satisfies conditions (i)–(iii) of Definition 1.1 with the equality (3) being replaced by the corresponding inequality

$$\int_Q \zeta_t \int_{v_0}^{v} h(r)\,db(r) + \zeta h(v)\,f \geqslant (\text{resp. } \leqslant) \int_Q a(v, Dv) \cdot D\zeta \tag{26}$$

satisfied for all non-negative $h \in C_c^1(\mathbb{R})$ and $\zeta \in \mathcal{D}([0, T[\times \Omega)$.

In generalization of Theorem 2.3 we have the following comparison result for sub- and supersolutions:

THEOREM 4.2. *Assume that* (H1)–(H2) *and the additional condition* (8) *on* a *hold. For* $i = 1, 2$, *let* $v_{0_i} \colon \Omega \to \overline{\mathbb{R}}$ *be measurable with* $b(v_{0_i}) \in L^1(\Omega)$, $f_i \in L^1(Q)$. *Let* v_1 *be a renormalized subsolution of* $(\mathrm{EP})(v_{0_1}, f_1)$, v_2 *a renor-*

malized supersolution of (EP)(v_{0_2}, f_2). *Then there exists* $\kappa \in \operatorname{sign}^+(v_1 - v_2)$ *such that, for a.e.* $t \in (0, T)$,

$$\int_\Omega (b(v_1)(t) - b(v_2)(t))^+ \leqslant \int_\Omega (b(v_{0_1}) - b(v_{0_2}))^+ + \int_0^t \int_\Omega \kappa (f_1 - f_2).$$

It is left to the reader to check that this result can be proved in the same way as Theorem 2.3 with the exception that, for sub/supersolutions, we cannot apply the "integration-by-parts-formula," Lemma 1.4. This difficulty is easily overcome by using the following more general

LEMMA 4.3. *Let* $b\colon \mathbb{R} \to \mathbb{R}$ *be continuous and non-decreasing with* $b(0) = 0$, $v \in L^p(0, T; W_0^{1,p}(\Omega))$ *with* $b(v) \in L^1(Q)$, $v_0\colon \Omega \to \overline{\mathbb{R}}$ *with* $b(v_0) \in L^1(\Omega)$. *Let* $G \in L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^1(Q)$ *and suppose that*

$$\int_Q \xi_t (b(v) - b(v_0)) \geqslant (\text{resp. } \leqslant) \int_0^T \langle G, \xi \rangle \, dt \tag{27}$$

for all non-negative $\xi \in \mathcal{D}([0, T[\times \Omega)$. *Then*

$$\int_Q \xi_t \int_{v_0}^v h(r) \, db(r) \geqslant (\text{resp. } \leqslant) \int_0^T \langle G, h(v) \xi \rangle \, dt \tag{28}$$

for all non-negative $h \in W^{1,\infty}(\mathbb{R})$ *and* $\xi \in \mathcal{D}([0, T[\times \mathbb{R}^N)$ *such that* $h(v)\xi \in L^p(0, T; W_0^{1,p}(\Omega))$.

A proof is given in the Appendix (see also [19]). Let us also mention other directions of possible extensions of our results. Note that results and proofs remain unchanged if the vector field a was allowed to depend on the time variable t as well. As to a possible dependance of a on the space variable x, the situation is more complicated. Consider a Caratheodory vector field $a\colon Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, i.e., $(t, x) \mapsto a(t, x, r, \xi)$ is measurable for all $(r, \xi) \in \mathbb{R} \times \mathbb{R}^N$, $(r, \xi) \mapsto a(t, x, r, \xi)$ is continuous for a.e. $(t, x) \in Q$ and assume that a satisfies a growth condition

$$|a(t, x, r, \xi)|^{p'} \leqslant C_1(t, x, |r|) + C_2(|r|)(1 + |\xi|^p), \tag{29}$$

where $C_1\colon Q \times \mathbb{R}^+ \to \mathbb{R}^+$ is Caratheodory, non-decreasing in $r \in \mathbb{R}^+$ with $C_1(\cdot, \cdot, r) \in L^1(Q)$ for all $r > 0$, $C_2\colon \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing. Assume, moreover, that a satisfies the condition

$$\begin{aligned}(a(t, x, r, \xi) - a(\tau, y, s, \eta))(\xi - \eta) + C(t, x, \tau, y, r, s)(1 + |\xi|^p + |\eta|^p)\, |r - s| \\ \geqslant (a_0(t, x, \xi) - a_0(\tau, y, \eta))(\xi - \eta) + \Gamma(t, x, \tau, y, r, s)\xi + \hat{\Gamma}(t, x, \tau, y, r, s)\eta\end{aligned}$$

with $a_0\colon Q\times\mathbb{R}^N\to\mathbb{R}^N$ a Caratheodory vector field, monotone in $\xi\in\mathbb{R}^N$, satisfying an appropriate growth condition as (29), $C\colon Q\times Q\times\mathbb{R}^2\to\mathbb{R}^+$ a Caratheodory function, monotone in $(r,s)\in\mathbb{R}\times\mathbb{R}$ with $C(\cdot,\cdot,\cdot,\cdot,r,s)\in L^\infty(Q\times Q)$ for all r,s and $\Gamma,\hat{\Gamma}\colon Q\times Q\times\mathbb{R}^2\to\mathbb{R}^N$ Caratheodory vector fields. If, moreover, we assume that

$$\operatorname{div}_x a_0(t,x,0)=0 \qquad \text{for a.e.} \quad t\in(0,T) \tag{30}$$

and

$$\operatorname{div}_x a(t,x,r,0)=0 \qquad \text{for all} \quad r\in\mathbb{R}, \quad \text{a.e. } t\in(0,T), \tag{31}$$

then it is possible to adapt the proof of Theorem 2.3 and we can show that the result of Theorem 2.3 (respectively Theorem 4.2) still holds. Condition (30) is essentially equivalent to the fact that the right hand side f is assumed to belong to $L^1(Q)$ (and not $L^1(Q)+L^{p'}(0,T;W^{-1,p'}(\Omega))$). However, condition (31) is a severe restriction which we suspect to be non necessary. If, for example, a is of the form $a=a_0(r,\xi)+P(t,x)\,f(r)$ with a_0 satisfying (8) and $P\colon Q\to\mathbb{R}^N$ measurable, $f\colon\mathbb{R}\to\mathbb{R}$ continuous, then (31) implies $\operatorname{div}_x P=0$. While there are interesting examples where this condition is satisfied (e.g., $P(t,x)=\nabla_x p(t,x)$ a vector field of gradients of pressure satisfying $\Delta_x p=0$ which is the case considered in [13, 16]), there are cases where this condition is not satisfied, but, nevertheless, uniqueness of solutions is expected to hold (cf. [2]).

Finally let us remark on the possibility of considering different boundary conditions as well as the Cauchy problem associated with the equation $b(v)_t=\operatorname{div} a(v,Dv)+f$ on the whole of $\mathbb{R}^N$. This will be considered in detail elsewhere.

APPENDIX

Proof of Lemma 4.3. Note first that $|\int_{v_0}^{v} h(r)\,db(r)|\leqslant\|h\|_\infty\,|b(v)-b(v_0)|$, hence $\int_{v_0}^{v} h(r)\,db(r)\in L^1(Q)$ and, in (28), each term is well-defined. Next, note that, if v satisfies one of the inequalities in (27), then $-v$ satisfies the other with b, v_0, G being replaced by $\tilde{b}(r)=-b(-r)$, $\tilde{v}_0=-v_0$ and $\tilde{G}=-G$, respectively. Therefore it will be sufficient to prove one of the two inequalities. So let us assume that $\int_Q \xi_t(b(v)-b(v_0))\geqslant\int_0^T\langle G,\xi\rangle\,dt$ for all non-negative $\xi\in\mathscr{D}([0,T[\times\Omega)$ (hence, by approximation, for all non-negative $\xi\in W^{1,\infty}(Q)\cap L^p(0,T;W_0^{1,p}(\Omega))$ with $\xi(T)=0$).

For the time being assume that h is non-decreasing. Let $\psi_h\colon\mathbb{R}\to\,]-\infty,\infty]$ be defined by $\psi_h(r)=\int_0^r h\circ(b^{-1})^0(s)\,ds$ for $r\in\overline{R(b)}$, $\psi_h(r)=+\infty$ otherwise, where, as usual, for any monotone graph $\beta\subset\mathbb{R}\times\mathbb{R}$ with $\beta(0)=0$, for any $s\in\mathbb{R}$, $\beta^0(s)$ denotes the element of minimal absolute value of $\beta(s)$. Note that ψ_h is a proper, l.s.c. convex function, $h \circ b^{-1} \subset \partial \psi_h$, i.e., $h(s) \in \partial \psi_h(b(s))$ for any $s \in \mathbb{R}$ and $\psi(b(r)) = \int_0^{b(r)} h \circ (b^{-1})^0 (s)\, ds = \int_0^r h(s)\, db(s)$. It is clear that $\int_r^{\hat{r}} h(s)\, db(s) = \psi_h(b(\hat{r})) - \psi_h(b(r)) \leqslant h(\hat{r}) \times (b(\hat{r}) - b(r))$ for any $r, \hat{r} \in \mathbb{R}$. Consequently, for any $\eta > 0$ and almost every $t > 0$,

$$(b(v(t)) - b(v(t-\eta)))\, h(v(t)) \geqslant \int_{v(t-\eta)}^{v(t)} h(r)\, db(r), \tag{32}$$

$$(b(v(t)) - b(v(t-\eta)))\, h(v(t-\eta)) \leqslant \int_{v(t-\eta)}^{v(t)} h(r)\, db(r) \tag{33}$$

almost everywhere in Ω where, for $t < 0$, $v(t) = v_0$. Let $\xi \in \mathscr{D}(\,]-\infty, T[\, \times \mathbb{R}^N)$, $\xi \geqslant 0$ with $h(v)\xi \in L^p(0, T; W_0^{1,p}(\Omega))$ and let $\zeta = h(v)\xi$. Note that, for any $\eta > 0$, the function $\zeta^\eta(t) = 1/\eta \int_t^{t+\eta} \zeta(s)\, ds \in W^{1,\infty}(Q) \cap L^p(0, T; W_0^{1,p}(\Omega))$, $\zeta^\eta(T) = 0$, and thus ζ^η is an admissible testfunction in (27). According to (32), using partial summation as in [1, 19], we find

$$\begin{aligned}
\int_0^T \langle G, \zeta^\eta \rangle\, dt &\leqslant \int_Q (\zeta^\eta)_t\, (b(v(t)) - b(v_0)) \\
&= \int_Q \frac{1}{\eta}\, (\zeta(t+\eta) - \zeta(t))(b(v(t)) - b(v_0)) \\
&= \int_Q \frac{1}{\eta}\, (b(v(t-\eta)) - b(v(t)))\, \zeta(t) \\
&= \int_Q \frac{1}{\eta}\, (b(v(t-\eta)) - b(v(t))\, h(v(t))\, \xi(t) \\
&\leqslant \int_Q \xi(t)\, \frac{1}{\eta} \int_{v(t)}^{v(t-\eta)} h(r)\, db(r) \\
&= \int_Q \int_{v_0}^{v(t)} h(r)\, db(r)\, \frac{1}{\eta}\, (\xi(t+\eta) - \xi(t)).
\end{aligned}$$

As $\zeta^\eta \to \zeta = h(v)\xi$ a.e. on Q and in $L^p(0, T; W_0^{1,p}(\Omega))$ and, moreover, remains uniformly bounded as $\eta \to 0$, passing to the limit with $\eta \to 0$ in the preceding inequality yields (28). Now suppose that h is non-increasing. Let $v_{0n} \in W_0^{1,p}(\Omega)$ with $b(v_{0n}) \to b(v_0)$ in $L^1(\Omega)$ as $n \to \infty$ and let n be fixed in the following. Using the same arguments as above (use (33) for $\tilde{h} = -h(r)$) yields

$$(b(v(t)) - b(v(t-\eta)))\, h(v(t-\eta)) \geqslant \int_{v(t-\eta)}^{v(t)} h(r)\, db(r) \tag{34}$$

for a.e. $t>0$, for any $\eta>0$, where, this time, for $t<0$, we define $v(t)=v_{0n}$. Let $\tilde{\zeta}=h(v)\xi$. Note that, if $h(0)\neq 0$, then the assumption $h(v)\xi\in L^p(0,T;W_0^{1,p}(\Omega))$ implies that $\xi\in L^p(0,T;W_0^{1,p}(\Omega))$. Consequently, in this case, we may assume that $\xi\in L^p(-T,T;W_0^{1,p}(\Omega))$. As a consequence, for any $\eta>0$, the function $\tilde{\zeta}^\eta(t)=1/\eta\int_{t-\eta}^t\tilde{\zeta}(s)\,ds$ belongs to $W^{1,\infty}(Q)\cap L^p(0,T;W_0^{1,p}(\Omega))$, $\tilde{\zeta}^\eta(T)=0$, hence $\tilde{\zeta}^\eta$ is admissible in (27). According to (34), using similar rearrangements as above, for η sufficiently small, we find

$$\begin{aligned}\int_0^T\langle G,\tilde{\zeta}^\eta\rangle\,dt&\leqslant\int_Q(\tilde{\zeta}^\eta)_t\,(b(v(t))-b(v_0))\\&=\int_Q\frac{1}{\eta}(\tilde{\zeta}(t)-\tilde{\zeta}(t-\eta))(b(v(t))-b(v_0))\\&=\int_0^T\int_\Omega\frac{1}{\eta}(b(v(t-\eta))-b(v(t)))\,\tilde{\zeta}(t-\eta)\\&\quad-\frac{1}{\eta}\int_0^\eta\int_\Omega\tilde{\zeta}(t-\eta)(b(v_{0n})-b(v_0))\\&\leqslant\int_Q\frac{1}{\eta}\xi(t-\eta)\int_{v(t)}^{v(t-\eta)}h(r)\,db(r)\\&\quad-\frac{1}{\eta}\int_{-\eta}^0\int_\Omega\xi(t)\,h(v_{0n})(b(v_{0n})-b(v_0))\\&=\int_Q\frac{1}{\eta}(\xi(t)-\xi(t-\eta))\int_{v_0}^{v(t)}h(r)\,db(r)\\&\quad+\frac{1}{\eta}\int_{-\eta}^0\int_\Omega\xi(t)\int_{v_0}^{v_{0n}}h(r)\,db(r)\\&\quad-\frac{1}{\eta}\int_{-\eta}^0\int_\Omega\xi(t)\,h(v_{0n})(b(v_{0n})-b(v_0)).\end{aligned}$$

Note that

$$\begin{aligned}\frac{1}{\eta}\int_{-\eta}^0\int_\Omega\xi(t)\int_{v_0}^{v_{0n}}h(r)\,db(r)&=\int_\Omega\xi(0)\int_{v_0}^{v_{0n}}h(r)\,db(r)\\&\quad+\frac{1}{\eta}\int_{-\eta}^0\int_\Omega(\xi(t)-\xi(0))\int_{v_0}^{v_{0n}}h(r)\,db(r),\end{aligned}$$

and the last integral converges to 0 as $\eta \to 0$. In the same way we have

$$\frac{1}{\eta}\int_{-\eta}^{0}\int_{\Omega}\xi(t)\,h(v_{0n})(b(v_{0n})-b(v_0))$$

$$=\int_{\Omega}\xi(0)\,h(v_{0n})(b(v_{0n})-b(v_0))$$

$$+\frac{1}{\eta}\int_{-\eta}^{0}\int_{\Omega}(\xi(t)-\xi(0))\,h(v_{0n})(b(v_{0n})-b(v_0)),$$

where again the last integral converges to 0 as $\eta \to 0$. Combining the last three estimates, using the fact that $\tilde{\zeta}^{\eta} \to h(v)\xi$ a.e. and in $L^p(0, T; W_0^{1,p}(\Omega))$ and remains uniformly bounded as $\eta \to 0$, we obtain

$$\int_0^T \langle G, h(v)\xi\rangle\, dt \leqslant \int_Q \xi_t \int_{v_0}^{v} h(r)\, db(r) + \int_{\Omega}\xi(0)\int_{v_0}^{v_{0n}} h(r)\, db(r)$$

$$-\int_{\Omega}\xi(0)\,h(v_{0n})(b(v_{0n})-b(v_0)).$$

As n is arbitrary, $b(v_{0n}) \to b(v_0)$ in $L^1(\Omega)$ and $|\int_{v_0}^{v_{0n}} h(r)\, db(r)| \leqslant \sup |h| \times |b(v_{0n})-b(v_0)|$, passing to the limit with $n \to \infty$ in the preceding inequality yields (28) in the case h is non-increasing. It follows that (28) is satisfied for any $h = h_1 \times h_2$ with h_1, h_2 monotone; indeed, it is sufficient to apply the preceding result for monotone functions two times: first with v, b and G for h_1, then with v, $B_{h_1}(r) = \int_0^r h_1(r)\, db(r)$ and $Gh_1(v)$ for h_2. As any $h \in W^{1,\infty}(\mathbb{R})$ may be approximated by convex combinations of these "product" functions, the assertion of the lemma follows. ∎

Proof of Lemma 1.4. This result is actually a simple consequence of Lemma 4.2. Indeed, according to the assumptions, v satisfies both inequalities in (27) with $G = -b(v)_t$ and, therefore, by the result of Lemma 4.2, we obtain an equality in (28) in the case of non-negative $h \in W^{1,\infty}(\mathbb{R})$, $\xi \in \mathcal{D}([0, T[\times \mathbb{R}^N)$, hence by approximation for all non-negative $\xi \in W^{1,\infty}(Q)$ with $\xi(T) = 0$ and $h(v)\xi \in L^p(0, T; W_0^{1,p}(\Omega))$. Due to the linearity in h and ζ the assertion of Lemma 1.4 follows. ∎

It remains to give the

Proof of Proposition 1.3. (i) Let v be a weak solution of (EP). In particular, $b(v)_t = \operatorname{div} a(v, Dv) + f$ in $\mathcal{D}'(Q)$, and thus $b(v)_t \in L^{p'}(0, T; W^{-1,p'}(\Omega))$

$+L^1(Q)$. Moreover, $b(v)(0,\cdot)=b(v_0)$ in $\mathscr{D}'(\Omega)$. Then, for $h\in C_c^1(\mathbb{R})$, $\xi\in\mathscr{D}([0,T[\times\Omega)$, by Lemma 1.4,

$$\int_0^T\int_\Omega \xi_t \int_{v_0}^{v} h(r)\,db(r) = -\int_0^T \langle b(v)_t(t), \xi(t)\,h(v(t))\rangle\,dt$$

$$= \iint a(v, Dv)\cdot D(h(v)\xi) - fh(v)\xi,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $W^{-1,p'}(\Omega)+L^1(\Omega)$ and $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. Next, for $n\in\mathbb{N}$, let $T_{n+1,n}=T_{n+1}-T_n$. Again by Lemma 1.4, we obtain

$$\int_\Omega\int_{v_0}^{v(t)} T_{n+1,n}(r)\,db(r) = \int_0^T \langle b(v)_t(t), T_{n+1,n}(v(t))\rangle\,dt$$

$$= \iint_{Q\cap\{n<|v|<n+1\}} a(v,Dv)\cdot Dv - \iint_Q fT_{n+1,n}(v)$$

$$= \iint_{Q\cap\{n<|v|<n+1\}} (a(v,Dv)-a(v,0))\cdot Dv$$

$$+\iint_{Q\cap\{n<|v|<n+1\}} a(v,0)\cdot Dv - \iint_Q fT_{n+1,n}(v). \tag{35}$$

As $|\int_{v_0}^{v} T_{n+1,n}(r)\,db(r)|\leqslant |b(v)-b(v_0)|\in L^1(Q)$ and $\int_{v_0}^{v} T_{n+1,n}(r)\,db(r)\to 0$ a.e. on Q as $n\to\infty$, by Lebesgue's dominated convergence theorem, the integral on the left of (35) converges to 0 as $n\to\infty$. Obviously, the same holds for the last integral on the right. Moreover, $\iint_{Q\cap\{n<|v|<n+1\}} a(v,0)\cdot Dv = \iint_Q \operatorname{div}\int_0^{T_{n+1}(v)} a(r,0)\,\chi_{\{|r|>n\}}\,dr = 0$. As a consequence, (4) holds and v is renormalized solution.

Conversely, assume that v is a renormalized solution. Applying (3) with $h(v)=H(n+1-|v|)$ where $H\in C^\infty(\mathbb{R})$, $H'\geqslant 0$, $H=0$ on $]-\infty,0]$, $H=1$ on $[1,\infty[$, we obtain (2) at the limit as $n\to\infty$ since

$$\int_{v_0}^{v} H(n+1-|r|)\,db(r)\to b(v)-b(v_0) \qquad \text{in } L^1(Q),$$

$$D(\xi H(n+1-|v|))\to D\xi \qquad \text{in } L^p(Q)^N.$$

Hence v is a weak solution of (EP).

(ii) Due to the growth condition (1) and the definition of a renormalized solution, (ii) is an immediate consequence of (i). ∎

REFERENCES

1. H. W. Alt and S. Luckhaus, Quasi-linear elliptic-parabolic differential equations, *Math. Z.* **183** (1983), 311–341.
2. S. J. Alvarez and J. Carrillo, A free boundary problem in theory of lubrication, *Comm. Partial Differential Equations* **19** (1994), 1743–1761.
3. Ph. Bénilan, L. Boccardo, Th. Gallouët, R. Gariepy, M. Pierre, and J.-L. Vazquez, An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **22** (1995), 241–273.
4. Ph. Bénilan and P. Wittbold, On mild and weak solutions of elliptic-parabolic problems, *Adv. Differential Equations* **1** (1996), 1053–1073.
5. Ph. Bénilan and P. Wittbold, Sur un problème parabolique–elliptique, *M2AN*, to appear.
6. D. Blanchard and F. Murat, Renormalised solutions of nonlinear parabolic problems with L^1 data: Existence and uniqueness, *Proc. Roy. Soc. Edinburgh Sect. A* **127** (1997), 1137–1152.
7. D. Blanchard and H. Redwane, Solutions rénormalisées d'équations paraboliques à deux non linéarités, *C.R. Acad. Sci. Paris* **319** (1994), 831–835.
8. L. Boccardo, D. Giachetti, J. I. Diaz, and F. Murat, Existence of a solution for a weaker form of a nonlinear elliptic equation, *in* "Recent Advances in Nonlinear Elliptic and Parabolic Problems," Pitman Research Notes in Mathematics Series, Vol. 208, pp. 229–246, Longman Sci. Tech., Harlow, 1989.
9. J. Carrillo, Unicité des solutions du type Kruskov pour des problèmes elliptiques avec des termes de transport non linéaires, *C.R. Acad. Sci. Paris* **303** (1986), 189–192.
10. J. Carrillo, On the uniqueness of the solution of the evolution dam problem, *Nonlinear Anal.* **22** (1994), 573–607.
11. J. Carrillo, Entropy solutions for nonlinear degenerate problems, submitted for publication.
12. J. Carrillo and M. Chipot, On some nonlinear elliptic equations involving derivatives of the nonlinearity, *Proc. Roy. Soc. Edinburgh* **100** (1985), 281–294.
13. A. Plouvier-Debaigt, B. Donne, G. Gagneux, and P. Urruty, Solutions rénormalisées pour des modèles des milieux poreux, *C.R. Acad. Sci. Paris Sér. I* **325** (1997), 1091–1095.
14. R. J. Di Perna and P. L. Lions, On the Cauchy problem for Boltzmann equations: Global existence and weak stability, *Ann. of Math. (2)* **130** (1989), 321–366.
15. G. Gagneux and M. Madaune-Tort, Unicité des solutions faibles d'équations de diffusion–convection, *C.R. Acad. Sci. Paris* **318** (1994), 919–924.
16. G. Gagneux and M. Madaune-Tort, "Analyse mathématique de modèles non linéaires de l'ingénierie pétrolière," Mathématiques et Applications, Vol. 22, Springer, Berlin, 1996.
17. S. N. Kruzhkov, First-order quasilinear equations in several independent variables, *Math. USSR Sb.* **10** (1970), 217–243.
18. F. Murat, "Soluciones renormalizadas de EDP elípticas no lineales," Publ. Laboratoire d'Analyse Numérique, Univ. Paris 6, 93023, 1993.
19. F. Otto, L^1-Contraction and uniqueness for quasilinear elliptic–parabolic problems, *J. Differential Equations* **131** (1996), 827–848.
20. J. M. Rakotoson, Generalized solutions in a new type of sets for problems with measures as data, *Differential Integral Equations* **6** (1993), 27–36.
21. J. M. Rakotoson, Uniqueness of renormalized solutions in a T-set for the L^1-data and the link between various formulations, *Indiana Univ. Math. J.* **43** (1994), 285–293.
22. P. Wittbold, Renormalized solutions of quasilinear elliptic–parabolic problems, in preparation.